RECITATION 6

EXTREME VALUE PROBLEMS

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Section 1. *Exercises*

Exercise 1

Find the critical points of e^x. Find the maximum value of e^x on $[0, \ln(5)]$.

Solution .:.

There are no critical points of e^x since its derivative (e^x) is never 0 and is always defined. The maximum value then just needs to be checked on the endpoints: $e^0 = 1$ while $e^{\ln(5)} = 5 > 1$. Hence 5 is the maximum value.

Exercise 2

Find the critical points of $f(x) = x^2 + 4x - 1$. Find the minimum and maximum value on $[-4, 4]$.

Solution .:.

The critical points are when the derivative $f'(x)$ is 0 or is undefined while $f(x)$ is defined. Note that $f'(x) = 2x + 4$ which is 0 iff $x = -2$. Since $f'(x)$ is always defined, this is the only critical point. We also have that $f(-4) = 16 - 16 - 1 = -1$, $f(-2) = 4 - 8 - 1 = -5$, $f(4) = 16 + 16 - 1 = 31$. Hence the minimum value is -5 and the max is 31.

Exercise 3

Find the minimum value of $f(x) = \ln x/x$ on $[1, e^{100}]$.

Solution .:.

Using the quotient rule, $f'(x) = \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \frac{1-\ln x}{x^2}$ this is 0 iff $\ln(x) = 1$ iff $x = e$. This is undefined for $x = 0$, but this isn't in the interval anyway. Testing the relevant points, we get that $f(1) = 0$, $f(e) = 1/e$ and $f(e^{100}) = 100/e^{100} < 1/e$ since $100 < 2^{99} < e^{99}$ (taking the reciprocals flips the inequalities) .

Exercise 4

Is there a minimum value of $2x + 4$ on $(1, 4]$? Is there a minimum value of $x^2 - 4x + 4$ on $(1, 4]$?

Solution .:.

No: for any x in $(1, 4]$, $x_1 = (1 - x)/2$ is also in $(1, 4]$ but $x_1 < x$ and so $2x_1 + 4 < 2x + 4$, implying that there can be no minimum value.
Yes: $x^2 - 4x + 4 = (x - 2)(x - 2)$ has a minimum value at $x = 2$ as a parabola.

Exercise 5

Find the extreme values of $f(x) = \sec(x)$ from $[\pi/4, \pi/3]$.

Solution .:.

$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\cos(x)} = \frac{\sin x}{\cos^2 x}$ this is 0 iff $\sin x = 0$ iff $x = \pi n$ for some integer n. This isn't in our interval, so we ignore these values. Similarly, $\cos(x) = 0$ iff $x = \pi/2 + \pi n$ for some integer n, which isn't in our interval:

again, we igore it. So there are no critical points within $[\pi/4, \pi/3]$. So to find the extreme values, we have $\sec(\pi/4) = \frac{1}{\cos(\pi/4)} = \frac{2}{\sqrt{2}} = \sqrt{2}$ and $\sec(\pi/3) = \frac{1}{\cos(\pi/3)} = \frac{1}{1/2} = 2$. So 2 is the maximum, and $\sqrt{2}$ is the minimum value.

Exercise 6

Write $f(x) = \ln(\sin x)$. Find the critical points of f within $[\pi/4, 3\pi/4]$.

Solution .:.

By the chain rule $f'(x) = \frac{1}{\sin x} \cdot \cos x = 0$ iff $\cos x = 0$ iff $x = \pi/2 + \pi n$ for integer n. The only one of those in our interval is $\pi/2$. $f'(x)$ is undefined iff $\sin x = 0$, which happens iff $x = \pi n$ for some integer n, which doesn't occur in our interval. So the only critical point in the interval is $\pi/2$.

Exercise 7

Write $f(x) = e^{x^2}$. Find the extreme values on $[-3, 3]$.

Proof .:.

$f'(x) = 2xe^{x^2} = 0$ iff $x = 0$. Since $f'(x)$ is defined everywhere, we have only one critical point: $x = 0$. Testing the relevant values, $f(-3) = f(3) = e^9$. $f(0) = e^0 = 1$ so the max value is e^9 and the min is 1.

Exercise 8

Write $f(x) = \frac{1}{x^2+1}$. Find the extreme values on $[-2, 2]$.

Proof .:.

$f'(x) = \frac{-2x}{(x^2+1)^2}$, which is 0 if an only if $-2x = 0$, meaning $x = 0$. $f'(x)$ is always defined since $x^2 \geq 0$ implies $x^2 + 1 > 0$ for all x (so the denominator of $f'(x)$ is never 0). So $x = 0$ is the only critical point. Evaluating at the endpoints of the interval and the critical point yields $f(-2) = \frac{1}{5} = f(2)$ while $f(0) = 1$. Hence the min value is $\frac{1}{5}$ and the max is 1.